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Duality on Compact Prime Ringed Spaces

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INTRODUCTION

How to describe the ring of global sections of a (compact) ringed space is an interesting question which has received much attention. For example, the ring of global sections of a compact local (resp. quasi-local) ringed space is precisely a Gelfand ring (resp. a strongly harmonic ring)—see [7, 8]; the ring of global sections of a (Stone) compact simple ringed space is precisely a biregular ring—see [3, Theorem I or 15, Theorem 2.6]; the ring of global sections of a Stone Hausdorff (not necessarily commutative) domain ringed space is precisely a Baer ring—see [4, Proposition 3.6]; we shall extend this result in this paper.

In this paper, we shall first investigate which rings can be expressed as the ring of global sections of a compact prime ringed space, which means a compact ringed space (in the sense of Mulvey [8]) of which each stalk is a prime ring.

One necessary and sufficient condition is that A satisfy (1) for each $a \in A$ the left annihilator $\mathrm{Ann}_l\langle a\rangle$ of the principal ideal generated by a is generated by a central idempotent element; and (2) for each minimal prime ideal P of $A, a \in P$ implies that $\mathrm{Ann}_l\langle a\rangle \not\subseteq P$. We call such rings *small weakly Baer rings*.

Then, we shall establish a duality between the category of compact prime ringed spaces and the category of small weakly Baer rings (see Theorem 3.9 below). As a consequence, we shall show that a compact prime ringed space is a Stone prime ringed space and, hence, that the category of compact prime ringed spaces is the category of Stone prime ringed spaces.

Using a similar argument, we finally establish a duality between the category of compact (not necessarily commutative) domain ringed spaces

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and the category of Baer rings (see Theorem 4.4 below), which extends the result of [4, Proposition 3.7] mentioned above.

For this purpose, we first extend several equivalences of commutative Baer rings (see [1, 5, 6]) to include our case, since these generalizations are essential for establishing our duality.

In this paper, we always assume that a ring has an identity but is not necessarily commutative and that an ideal means a 2-sided ideal. The set of all 2-sided ideals of R will be denoted by Id R. The prime radical of an ideal I (i.e., the intersection of all prime ideals containing I) will be denoted by $\sqrt{}I$ (or $\sqrt{I}$).

1. Compact Prime Ringed Spaces

In this section, we would like to give some characterizations of the ring of global sections of a compact prime ringed space.

Definition 1.1. A ringed space $(X, \mathscr{F})$ is called *compact prime* if X is compact, if for any distinct x_1 and x_2 of X there is a global section ρ such that $\rho(x_1) = 1_{x_1}$ and $\rho(x_2) = 0_{x_2}$, and if each stalk of $\mathscr{F}$ is a prime ring (i.e., the zero ideal is prime).

Let A be the ring of global sections of $(X, \mathscr{F})$ and let $I_x = \{\rho \in A \mid \rho(x) = 0\}$ for each $x \in X$.

Proposition 1.2. *Let $(X, \mathscr{F})$ be a compact prime ringed space. Then the ring A of global sections of $(X, \mathscr{F})$ has the following properties*:

(1) *The minimal prime ideal space* MinSpec A *of A is compact.*

(2) *Each prime ideal of A contains a unique minimal prime ideal.*

(3) *A is semiprime* (*i.e.*, $0 = \sqrt{}0$).

(4) *For each minimal prime ideal P of A and for each $a \in P$,* $\mathrm{Ann}_l\langle a\rangle \not\subseteq P$.

Proof. Since each compact ringed space is completely regular (see [15, Lemma 2.2]), by [8, Theorem 1.4] we see that each stalk $\mathscr{F}_x$ of $\mathscr{F}$ at x is isomorphic to A/I_x. Thus each I_x is a prime ideal of A since each A/I_x is a prime ring. Then by [15, Lemma 3], each prime ideal of A contains I_x for some $x \in X$, and in fact for a unique $x \in X$ by the compactness. Thus A satisfies (2). Moreover, the minimal prime ideal space of A is the space $\{I_x | x \in X\}$, which is homeomorphic to X (see also [8]) and hence is compact; that is, A satisfies (1). Since $\bigcap I_x = 0$, A is semiprime. For each

I_x and for each $\sigma \in I_x$, there is an open neighbourhood V_x of x such that σ vanishes on V_x; and hence we can define a section ψ which is 1 at x and whose support lies in V_x. Thus $\psi A \sigma = 0$ but $\psi \notin I_x$; that is, A satisfies (4) (for the details see [15, Lemma 4]).

DEFINITION 1.3. For any ideal I of a ring R, let $I^{\perp} = \Sigma\{J \mid JI \subseteq \sqrt{0}\}$. An ideal I of R is called *small* if $a \in I$ implies $\langle a \rangle^{\perp} \not\subseteq I$. A ring R is called *small* if each minimal prime ideal of R is small.

Thus the properties (3) and (4) in Proposition 1.2 precisely express that the ring of global sections of a compact prime ringed space is small and semiprime. The class of small rings includes the following rings (for details see [15]):

(1) All qc-rings in the sense of [2]; in particular, all commutative rings, all biregular rings, and all (right) Noetherian rings.

(2) All weakly symmetric rings in the sense of [14]; in particular, all rings without non-zero nilpotents.

(3) All prime rings; in particular, all free rings.

We shall show below that a ring which has the four properties in Proposition 1.2 can be expressed as the ring of global sections of a compact prime ringed space.

2. WEAKLY BAER RINGS

In this section, we will give some equivalent conditions of those rings which have the properties described in Proposition 1.2. These results will be used in Section 3. Following Hofmann, a ring R is called *Baer* if for any $a \in R$, the set $\{b \in R \mid ba = 0\}$ is an ideal generated by a central idempotent element; note that this notion is close to, but is different from, that of Kalpansky. A ring R is called *biregular* if each principal ideal is generated by a central idempotent. We would like to introduce the following.

DEFINITION 2.1. A ring R is called *weakly Baer* if each left annihilator $\mathrm{Ann}_l\langle a \rangle$ is generated by a central idempotent element of R.

It is clear that both biregular rings and Baer rings are weakly Baer (see also Section 4). We shall show below that a ring R is Baer iff R is weakly Baer and has no non-zero nilpotents (see Lemma 4.1 below). Moreover, we shall see below that both biregular rings and Baer rings are small; and, hence, both of them are small weakly Baer rings. Note that the commutative weakly Baer rings are precisely the commutative Baer rings.

Commutative Baer rings have received much attention and several equivalent conditions of them have been given. We summarize the following equivalences, which appeared in [1, 5, 6, 11, 13] and so on. For a commutative ring, the following are equivalent:

(a) R is Baer.

(b) The restriction from MinSpec R to Spec BR is a homeomorphism, where BR is the Boolean algebra consisting of all idempotents of R.

(c) MinSpec R is a compact space and each prime ideal contains a unique minimal prime ideal.

(d) MinSpec R is a compact space and Id R is co-normal.

(e) There is a continuous retraction Spec $R \to$ MinSpec R.

In this section, we shall extend these equivalences to include our small weakly Baer rings. This generalization is needed to establish our duality (see Section 3). To prove it, we need several notations and notions.

For a ring R, we write $L(R)$ for the distributive lattice generated by the $\sqrt{\langle x \rangle}$ with $x \in R$. Then the typical element in $L(R)$ is of the form $\sqrt{I_1 I_2 \dots I_n}$, where I_j are finitely generated ideals of R.

Let R^* denote the set of all ideals having the form $I_1 I_2 \dots I_n$, where each I_i is finitely generated; then $L(R) = \{\sqrt{K} | K \in R^*\}$.

Recall that a distributive lattice L is called *Stonian* if for each $a \in L$ there exists a complemented element $b \in L$ such that b is the greatest element with the property that $b \wedge a = 0$.

First we have the following (see [16, Theorem 1.9]).

THEOREM 2.2. *A ring R is weakly Baer iff $L(R)$ is Stonian and R is semiprime.*

DEFINITION. For each $P \in$ Spec R, define $m(P) = \{K \in L(R) | K \subseteq P\}$.

Then we have the following.

LEMMA 2.3 [15]. (1) *The map m*: Spec $R \to$ Spec $L(R)$ *is an embedding.*

(2) *m is a homeomorphism between* MaxSpec R *and* MaxSpec $L(R)$.

LEMMA 2.4. *For any ring R, m*(MinSpec R) *is a dense subspace of* Spec $L(R)$.

Proof. Let $Q \in \operatorname{Spec} L(R)$ with $Q \not\supseteq K$ for some $\sqrt{K} \in L(R)$. Then $\sqrt{K} \neq \sqrt{0}$ and so there is a minimal prime ideal P such that $P \not\supseteq K$, since the intersection of all minimal prime ideals of R is $\sqrt{0}$. Thus, $\sqrt{K} \notin m(P)$; that is, $m(P)$ is in the open subset, determined by K, of $\operatorname{Spec} L(R)$.

In [15], we showed that R is small iff $m(\operatorname{MinSpec} R) \subseteq \operatorname{MinSpec} L(R)$. Thus we have the following.

LEMMA 2.5. *If R is small, then $m(\operatorname{MinSpec} R)$ is a dense subspace of* $\operatorname{MinSpec} L(R)$.

Recall that MinSpec R has a basis consisting of the

$$d(a) = \{P \in \operatorname{MinSpec} R | a \notin P\},$$

with $a \in R$.

LEMMA 2.6. *If R is a small weakly Baer ring then* MinSpec R *is compact zero-dimensional.*

Proof. First we show that MinSpec R has a basis consisting of the $d(e)$, where e is a central idempotent element. In fact, let $P \in d(a)$; then $a \notin P$ and hence $\langle e \rangle = \operatorname{Ann}\langle a \rangle \subseteq P$, where e is central idempotent. Now we can easily check that $P \in d(1-e) \subseteq d(a)$ by using smallness.

Now it suffices to prove that if $\{d_{e_i}\}$ is a open cover of MinSpec R, where e_i are central idempotents, then $\Sigma\langle e_i \rangle = R$. If not, there is a maximal ideal M containing all e_i. Let P be a minimal prime ideal contained in M. Then $P \in d_{e_i}$ for some i and, hence, $1 - e_i \in P$. Thus $1 = e_i + (1 - e_i) \in M$—a contradiction.

Combining this with Lemma 2.5, we have the following.

COROLLARY 2.7. *If R is a small weakly Baer ring, then the map m gives a homeomorphism between* MinSpec R *and* MinSpec $L(R)$.

Recall that a distributive lattice L is called *co-normal* if for all $a, b \in L$ with $a \wedge b = 0$ there exists $c, d \in L$ such that $c \wedge a = 0 = d \wedge b$ and $c \vee d = 1$. Similarly, we define a ring R to be *co-normal* if for all $I, J \in R^*$ with $IJ = 0$ (or, equivalently, for all finitely generated ideals K_1, K_2) there are $a, b \in R$ such that $a + b = 1$ and $aI = 0 = bJ$ (or, equivalently, $aK_1 = 0 = bK_2$ and $a + b = 1$). It is clear that R is co-normal iff $L(R)$ is co-normal.

To show Theorem 2.8 below, we need the following known result.

LEMMA [13, Theorem 5.1]. *For a distributive lattice L, the following are equivalent*:

(1) *L is Stonian.*

(2) *The restriction map* MinSpec $L \to$ Spec BL *is a homeomorphism.*

(3) *The space* MinSpec L *is compact and L is co-normal.*

(4) *There is a continuous retraction* Spec $L \to$ MinSpec L.

Now we can prove our main result.

THEOREM 2.8. *Let R be a small semiprime ring. Then the following are equivalent*:

(a) *R is weakly Baer.*

(b) *The restriction from* MinSpec R *to* Spec BR *is a homeomorphism, where BR is the Boolean algebra consisting of all central idempotents of R.*

(c) MinSpec R *is a compact space and each prime ideal contains a unique minimal prime ideal.*

(d) MinSpec R *is a compact space and* Id R *is co-normal.*

(e) *There is a continuous retraction* Spec $R \to$ MinSpec R.

Proof. (a) $\Rightarrow$ (b) By Corollary 2.7, we see that m is a homeomorphism from MinSpec R to MinSpec $L(R)$ and that $L(R)$ is a Stonian lattice; and, hence, the conclusion follows from the fact that MinSpec $L(R)$ is homeomorphic to Spec $B(L(R)$ and the fact that $B(L(R))$ is isomorphic to BR.

(b) $\Rightarrow$ (a) Note that condition (b) implies that MinSpec R is compact. Therefore m(MinSpec R) is a closed subspace of MinSpec $L(R)$ since MinSpec $L(R)$ is zero-dimensional and, hence, is the whole MinSpec $L(R)$ by Lemma 2.5. Thus the two spaces are homeomorphic, whence the restriction from MinSpec $L(R)$ to Spec $BL(R)$ is a homeomorphism. So $L(R)$ is a Stonian lattice and R is weakly Baer.

(a) $\Rightarrow$ (d) Let K_1, K_2 be finitely generated ideals of R with $K_1K_2 = 0$. Then, we have that Ann K_1 is a principal ideal generated by a central idempotent element, say e. Therefore $\langle e\rangle K_1 = 0 = \langle 1 - e\rangle K_2$ and $e + (1 - e) = 1$.

(d) $\Rightarrow$ (c) Let P_1 and P_2 be distinct minimal prime ideals of R. Since MinSpec R is T_2, there are $a, b \in R$ such that the open subsets determined by $\langle a\rangle$ and $\langle b\rangle$ are disjoint, whence $\langle a\rangle\langle b\rangle = 0$ since R is semiprime. Furthermore, since Id R is co-normal, there are $x, y \in R$ such that $\langle a\rangle\langle x\rangle = 0 = \langle b\rangle\langle y\rangle$ and $x + y = 1$. Thus $P_1 + P_2 = R$ since $x \in P_1$ and $y \in P_2$.

(c) $\Rightarrow$ (a) Since MinSpec R is compact, m is homeomorphic to MinSpec $L(R)$. Let P be a prime ideal of $L(R)$. Then there is a maximal ideal $M \supseteq P$. Since $M \in m(\text{MaxSpec } R)$, M contains a unique minimal prime ideal of $L(R)$ by the assumption and the fact that $m(\text{MinSpec } R) =$ MinSpec $L(R)$. Thus P contains a unique minimal prime ideal of $L(R)$ and hence $L(R)$ is a Stonian lattice and hence R is weakly Baer.

(a) $\Rightarrow$ (e) Since $L(R)$ is Stonian, MinSpec $L(R)$ is a retract of Spec $L(R)$. So the conclusion follows from the facts that $m(\text{Spec } R) \subseteq \text{Spec } L(R)$ and that $m(\text{MinSpec } R) = \text{MinSpec } L(R)$.

(e) $\Rightarrow$ (c) The compactness of MinSpec R follows from the compactness of Spec R. To show that each prime ideal contains a unique minimal prime ideal, let n be the retraction from Spec R to MinSpec R. For each prime ideal P of R, let P_0 be a minimal prime ideal contained in P. If $P_0 \neq n(P)$, then $n(P) \in d(P_0)$. Thus $P \in n^{-1}(d(P_0))$ and hence $P_0 \in n^{-1}d(P_0)$ and $P_0 = n(P_0) \in d(P_0)$—a contradiction.

Combining this with Proposition 1.2, we have the following.

COROLLARY 2.9. *Let $(X, \mathscr{F})$ be a compact prime ringed space. Then the ring of global sections of $(X, \mathscr{F})$ is a small weakly Baer ring.*

3. SHEAF STRUCTURES

In this section we shall construct, for a given small weakly Baer ring R, a compact prime ringed space such that R is isomorphic to the ring of global sections of this ringed space.

Let R be a ring. Define $O(P) = \sum\{\text{Ann}_l\langle a\rangle \mid a \notin P\}$. Then $O(P)$ is an ideal contained in P if P is a prime ideal.

We are now going to construct a sheaf space over $X = \text{MinSpec } R$ when R is a weakly Baer ring; in fact, we shall do more.

Recall that the typical open set of MinSpec R is the set $d_I = \{P \in X \mid I \subseteq P\}$ and that $\{d(a) \mid a \in R\}$ is a basis. Let q_P denote the canonical quotient map from R to $R/O(P)$ and let E be the disjoint union of the $R/O(P)$ for $P \in X$. Each $a \in R$ determines a Gelfand function $\hat{a}\colon X \to E$ by defining $\hat{a}(P) = q_P(a)$ for $P \in X$.

We endow E with the finest topology for which all the maps $\hat{a}$ are continuous and let $\eta\colon E \to X$ denote the canonical projection.

LEMMA 3.1. *The sets of the form $\hat{a}(d(b))$ for $a, b \in R$ form a basis of the topology on E and (E, η) is a sheaf space over X. For each $a \in R$ the function $\hat{a}$ is a global section and $a \mapsto \hat{a}$ is a ring homeomorphism.*

Proof. First we show that the set $V = \{P \in X | a - b \in O(P)\}$ is open for all $a, b \in R$. If $P \in V$, then $a\text{-}b \in \operatorname{Ann}_l\langle x\rangle$ for some $x \notin P$. Now $P \in d(x) \subseteq V$, as is easily seen. Next we show that

$$\hat{b}^{-1}(\hat{a}(d(x))) = \{P \in X | a - b \in O(P)\} \cap d(x)$$

for each $a, b \in A$, for $P \in \hat{b}^{-1}(\hat{a}(d(x)))$ if and only if $\hat{b}(P) = \hat{a}(P')$ for some $P' \in d(x)$, but this occurs if and only if $a - b \in P$ and $P = P' \in d(x)$. The rest of the proof is straightforward.

THEOREM 3.2. *Let R be a semiprime ring such that, if $\{d_{I_i}\}$ is an open cover of the space* MinSpec R, *then $\Sigma \bar{I}_i = R$, where $\bar{I}$ is the greatest ideal J with $d_J = d_I$. Then $a \mapsto \hat{a}$ is an isomorphism from R onto the ring A of all sections of the sheaf space (E, η) over* MinSpec R.

Proof. The injectivity follows from the semiprimeness of R. It remains to show the surjectivity. Let $\sigma \in A$. For each $P \in$ MinSpec R, there is an $a_P \in A$ with $\sigma(P) = \hat{a}_P(P)$. Then $\sigma^{-1}(\hat{a}_P(X)$ is a neighbourhood of P which contains some neighbourhood $d(I_P)$ of P, so that $\sigma | d(I_P) = \hat{a}_P | d(I_P)$ since they are sections. Thus $\{d(I_P) | P \in X\}$ is an open cover of MinSpec R and hence $\Sigma I_P = R$ by assumption. So there is a finite $x_i \in I_{P_i}$ such that $\Sigma x_i = 1$. Note that we also have $\sigma | d(x_i) = \hat{a}_i | d(x_i)$ for all $1 \leq i \leq n$.

Now let $a = \Sigma_1^n a_i x_i$. We claim that $\hat{a} = \sigma$. In fact, for each $i \leq n$ and each $P \in$ MinSpec R, if $P \notin d(x_i)$ (i.e., if $x_i \in P$), then $[(\sigma - \hat{a}_i)\hat{x}_i](P) \subseteq \hat{x}_i(P) \in P/P = 0$ by Lemma 3.3 below; while if $P \in d(x_i)$, then $\sigma(P) = \hat{a}_i(P)$, which implies that $[(\sigma - \hat{a}_i)\hat{x}_i](P) = 0$. Thus we have shown that $\Sigma_i^n[(\sigma - \hat{a}_i)\hat{x}_i](P) = 0$, that is, that $\hat{a} = \sigma$.

The following lemma is straightforward.

LEMMA 3.3. *Let R be a small semiprime ring and P a minimal prime ideal of R. Then $O(P) = P$.*

COROLLARY 3.4. *Let R be a small semiprime ring such that whenever $\{d_{I_i}\}$ is an open cover of the space* MinSpec R, *we have $\Sigma \bar{I}_i = R$. Then R has a sheaf representation of which each stalk is a prime ring and the base space is a zero-dimensional space.*

By the proof of Lemma 2.6, we see that a small weakly Baer ring R satisfies the condition in Theorem 3.2 and that MinSpec R is a compact zero-dimensional space. So we have the following.

THEOREM 3.5. *Each small weakly Baer ring R can be expressed as the ring of global sections of a Stone prime ringed space $(X, \mathcal{F})$, where X is* MinSpec R *and each stalk is isomorphic to R/P for a minimal prime ideal of R.*

We shall show below that, for small weakly Baer rings, Simmons representation theorem, Pierce representation theorem, and Theorem 3.2 agree, while the last equivalence makes our representation functorial and, hence, determines a duality.

LEMMA 3.6. *Let R be a small weakly Baer ring. Then P is a minimal ideal of R if and only if $P = O(M)$ for some $M \in$ Max R.*

Proof. Let P be a minimal prime ideal of R. Then there is a maximal ideal M containing P and, hence, $O(M) \subseteq P$. If $a \in P$ but $a \notin O(M)$ we have $\langle e \rangle = \mathrm{Ann}_l\langle a \rangle \not\subseteq P$ since P is small; but $e \in M$ since $a \notin O(M)$. On the other hand, $1 - e \in P$ since P is prime and, hence, $1 \in M$—a contradiction.

In fact, we have shown that if $P \subseteq M$, where M is maximal, then $P = O(M)$; it follows that each $O(M)$ is a minimal prime ideal of R.

Let R be a ring. Recall that $I \in IdR$ is called *uniformly virginal* if for each $a \in I$, we have

$$I + \mathrm{Ann}_l(Ra) = R,$$

where $\mathrm{Ann}_l(X)$ denotes the left annihilator of X. We denote by ΨR the set of all such ideals. It was shown by Simmons [12, Theorem 2.4] that ΨR is a subframe of Id R. For each $I \in$ Id R, we write Wir(I) the greatest uniformly virginal ideal contained in I.

Let $pt(\Psi R)$ denote the set of prime elements of ΨR and let (E_1, η_1) denote the sheaf space, where E_1 is the disjoint union of the quotient rings $R/\mathrm{Wir}(M)$, $M \in$ Max R, with the finest topology such that all $\hat{r}$ with $r \in R$ are continuous. Then $a \mapsto \hat{a}$ is an isomorphism from R onto the ring A of all global sections of the sheaf space over the space $\{\mathrm{Wir}(M) | M \in \mathrm{MaxSpec}\, R\}$ (for details see [12]).

LEMMA 3.7. *Let R be a small weakly Baer ring. Then $O(M) = \mathrm{Wir}(M)$ for each maximal ideal M of R; and hence $\{\mathrm{Wir}(M) | M \in \mathrm{MaxSpec}\, R\}$ is homeomorphic to* MinSpec R.

Proof. In fact it is always true that $\mathrm{Wir}(M) \subseteq O(M)$. Now let $a \in O(M)$. Then $\langle e \rangle = \mathrm{Ann}_l\langle a \rangle \not\subseteq M$ and, hence, $\mathrm{Ann}\langle e \rangle \subseteq O(M)$, so that $\langle e \rangle + O(M) = R$. Thus $O(M)$ is uniformly virginal and, hence, $O(M) \subseteq \mathrm{Wir}(M)$.

LEMMA 3.8. *Let R be a small weakly Baer ring. Then QR, for each prime ideal Q of BR, is a minimal prime ideal of R, so that the Pierce representation and Theorem* 3.2 *agree.*

Proof. Let $Q \in \mathrm{Spec}\, BR$ and let M be a maximal ideal containing QR. Then $QR \subseteq O(M)$. By Lemma 3.6, it suffices to show that $QR =$

$O(M)$. Let $a \in O(M)$. Then there is a central idempotent $e \in M$ such that $a \in \langle e \rangle$. But $M \cap BR = Q$ since Q is a maximal ideal of BR and, hence, $e \in QR$ so that $a \in QR$.

We have shown that any small weakly Baer ring R is associated with a Stone prime ringed space $(X, \mathscr{F})$, where X is isomorphic to Spec BR and each stalk is isomorphic to R/QR for some $Q \in$ Spec BR.

Recall that a ringed space $(X, \mathscr{F})$ is called reduced if X is a Stone space and if for each central idempotent global section σ and each $x \in X$ we have either $\sigma(x) = 0_x$ or $\sigma(x) = 1_x$. Pierce showed in [9] that each ringed space (Spec $BR, \mathscr{F}R$), where $\mathscr{F}R$ is the Pierce sheaf, is reduced. Thus, by Propositions 1.2 and 3.8, each compact prime ringed space is reduced. An alternative direct proof is the following: recall that a ring is called indecomposable if R has no trivial central idempotents. It is easy to check that a ringed space of which each stalk is indecomposable is reduced. Now the conclusion follows from the fact that each prime ring is trivially indecomposable.

Recall that a morphism Ψ from a ringed space $(X, \mathscr{F}_1)$ to a ringed space $(Y, \mathscr{F}_2)$ is a pair consisting of a continuous mapping f of X into Y and a continuous mapping ψ: $X +_f \mathscr{F}_1 \to \mathscr{F}_2$, where $X +_f \mathscr{F}_1$ is the pullback of X and $\mathscr{F}_1$, such that each $\psi(x, -)$ is a homeomorphism of $\mathscr{F}_{1, f(x)}$ to $\mathscr{F}_{2, x}$.

Let R_1 and R_2 be two small weakly Baer rings and g: $R_1 \to R_2$ a ring homomorphism preserving central idempotents. Then g induces a map from the Boolean algebra BR_1 to BR_2 and, hence, induces a continuous mapping g^* from Spec BR_2 to Spec BR_1 and a continuous mapping from MinSpec R_2 to MinSpec R_1. Pierce showed that this g^* also induces a morphism from (Spec $BR_2, \mathscr{F}R_2$) to (Spec $BR_1, \mathscr{F}R_1$) and, hence, determines a functor from the category of small weakly Baer rings and their morphisms to the category of reduced ringed spaces, which has an inverse (that is, which determines a duality between these categories). By Theorems 2.10 and 3.8, we have the following.

THEOREM 3.9. *The category of small weakly Baer rings and their morphisms is dual to the category of compact prime ringed spaces.*

4. APPLICATIONS

Let R be a ring and let BR be the Boolean algebra consisting of the central idempotents of R. Recall that a ring R is Baer iff for each $a \in R$ there is a central idempotent $e \in BR$ such that $ae = 0$ and such that if $ba = ab = 0$ then $b = be$.

As mentioned in the introduction, Preecy and Hofmann showed in [4, Proposition 3.7] that R is a Baer ring iff its Pierce sheaf is a Hausdorff sheaf of (not necessarily commutative) domains over a Stone space. We shall extend the above result by showing that there is a duality between the category of Baer rings and their morphisms and the category of compact domain ringed spaces, using our Theorem 3.9. First we need a lemma.

LEMMA 4.1. *A ring R is Baer iff R is weakly Baer without non-zero nilpotents.*

Proof. Let R be a Baer ring. Then R has no non-zero nilpotents and hence $\operatorname{Ann} a = \operatorname{Ann}\langle a \rangle$. So R is weakly Baer. The converse follows from the fact $\operatorname{Ann} a = \operatorname{Ann}\langle a \rangle$.

PROPOSITION 4.2. *The ring of global sections of a Pierce sheaf of domains over a Stone space is a Baer ring.*

Proof. By Theorem 3.9, we see that the ring A of global sections of a Pierce sheaf of domains over a Stone space is a weakly Baer ring. It suffices to show that A has no non-zero nilpotents. Let $\sigma^2 = 0$. Then for each minimal prime ideal P we have $(\sigma(P))^2 = 0_P$ and, hence, $\sigma(P) = 0$; that is, $\sigma \in P$. Thus $\sigma \in \cap\{P | P \in \operatorname{MinSpec} R\} = 0$.

THEOREM 4.3. *For a ringed space $(X, \mathcal{F})$, the following are equivalent:*

(1) *X is a Stone space and each stalk is a Hausdorff domain.*

(2) *X is a Stone space and each stalk is a domain.*

(3) *$(X, \mathcal{F})$ is a compact ringed space space and each stalk has no non-zero nilpotents.*

(4) *The ring of global sections of $(X, \mathcal{F})$ is a Baer ring.*

COROLLARY. *The category of compact domain ringed spaces is the category of Stone domain spaces.*

THEOREM 4.4. *The category of Baer rings and their morphisms is dual to the category of compact domain ringed spaces.*

Now let X be a nonempty set and let Z be the ring of integers. Then it is easy to see that $A = Z^X$ is a Baer ring (see also [5]). In [10], Scott showed that there is a bijection of the set of minimal prime ideals of A upon the set of ultrafilters on X. This result also follows easily from our Theorem 2.10; note that the Boolean algebra EA is isomorphic to $\mathcal{P}(X)$, the power set of X. We see that $\operatorname{MinSpec} A$ is homeomorphic to $\operatorname{Spec} \mathcal{P}(X)$, whose base set is the set of all ultrafilters on X. In fact, we have the following more general result.

PROPOSITION 4.5. *Let R be a not necessarily commutative domain (i.e., $ab = 0$ implies that $a = 0$ or $b = 0$) and let X be a set. Then $A = R^X$ is a Baer ring without non-zero nilpotents.*

Proof. Let $f \in A$. Then $g \in \operatorname{Ann} f$ iff $g^{-1}(0) \cup f^{-1}(0) = X$. Let e be the characteristic function of the set $X \setminus f^{-1}(0)$ (i.e., $e(x) = 1$ for $x \notin f^{-1}(0)$ and $e(x) = 0$ for $x \in f^{-1}(0)$). Then e is a central idempotent element of A and $\operatorname{Ann} f$ is generated by e.

Thus we can extend the above result of Scott in the following way.

PROPOSITION 4.6. *Let A be the ring defined in Proposition 4.5. Then there is a bijection of the set of minimal prime ideals of A upon the set of ultrafilters on X.*

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